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# *On the Order of Terms in a Semi-Convergent Series.*

BY HENRY P. MANNING.

We may give to a semi-convergent series of real terms any value we please by suitably changing the order of its terms. (See Jordan, Cours d'Analyse, t. I, 1893, p. 277.)

Suppose we have a series of the form

$$S = f(1) - f(2) + \dots + f(2n - 1) - f(2n) + \dots,$$

where  $f(x)$  is a continuous positive function, at least for large values of  $x$ .

We have

$$\lim f(n) = 0 \quad \text{and} \quad \lim \frac{f(n+1)}{f(n)} = 1,$$

but

$$f(1) + f(3) + \dots + f(2n - 1) + \dots,$$

and

$$f(2) + f(4) + \dots + f(2n) \quad + \dots,$$

both divergent series.

Suppose for all values of  $r$  greater than  $n$

$$\frac{f(r+\theta)}{f(r)} > m \quad \text{and} \quad < M \quad (0 \leq \theta \leq 2),$$

then

$$\int_0^2 f(r+\theta) d\theta > \int_0^2 m f(r) d\theta$$

and

$$< \int_0^2 M f(r) d\theta,$$

or

$$f(r) < \frac{1}{2m} \int_r^{r+2} f(x) dx,$$

and

$$> \frac{1}{2M} \int_r^{r+2} f(x) dx.$$

$m$  and  $M$  are functions of  $n$ , and for  $n$  indefinitely large they will usually have the same limit, 1.

Now in the series  $S$  we will take  $n'$  positive terms and  $n$  negative terms,  $n' > n$ , and consider the sum

$$S'_{n'+n} = f(1) + f(3) + \dots + f(2n' - 1) \\ - f(2) - f(4) - \dots - f(2n),$$

or 
$$S'_{n'+n} = S_{2n} + f(2n + 1) + f(2n + 3) + \dots + f(2n' - 1).$$

[If  $n' < n$ , these terms will be negative.]

$$\therefore S'_{n'+n} < S_{2n} + \frac{1}{2m} A,$$

and

$$> S_{2n} + \frac{1}{2M} A,$$

where

$$A = \int_{2n+1}^{2n'+1} f(x) dx. \quad (1)$$

There will be a limit if  $A$  has a limit, and we shall generally have

$$S' = S + \frac{1}{2} \lim A.$$

Since the series  $S$  is semi-convergent,

$$\lim \int_a^n f(x) dx = \infty, \text{ and } \lim \int_n^{n+a} f(x) dx = 0,$$

$a$  being a finite quantity.

Equation (1) establishes a relation between  $n'$  and  $n$  if we assign some value to  $A$ ; or establishes the value of  $A$  corresponding to some relation connecting  $n'$  and  $n$ .

This relation will reduce approximately for large values of  $n$  to a simple form satisfied by integer values of  $n'$  and  $n$ .

If these solutions are

$$n' = n_1, n_3, \dots, n_{2r-1}, \dots, \\ n = n_2, n_4, \dots, n_{2r}, \dots,$$

and if we write

$$\alpha_{2r-1} = f(2n_{2r-3} + 1) + f(2n_{2r-3} + 3) + \dots + f(2n_{2r-1} - 1), \\ \alpha_{2r} = f(2n_{2r-2} + 2) + f(2n_{2r-2} + 4) + \dots + f(2n_{2r}),$$

we have

$$S' = (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4) + \dots + (\alpha_{2r-1} - \alpha_{2r}) + \dots \\ = S + \frac{1}{2} \lim A.$$

The parentheses may be removed if the  $\alpha$ 's tend to zero as a limit, and in any case we can arrange the order of the terms in the group  $\alpha_{2r-1} - \alpha_{2r}$  in such

a way that the parenthesis will not be necessary. See second example discussed below.

$S'$ , then, will be a series whose terms are the terms of the series  $S$  arranged in a different order, and we see how we can actually change the order of the terms of a given semi-convergent series, if it can be expressed as assumed above, so that it will take any value we please; and, conversely, we have a method of getting the value of the new series  $S'$  produced from  $S$  by a given change in the order of the terms, if the change can be expressed by a relation between  $n'$  and  $n$  as assumed.

Take for example the series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

Here (1) becomes

$$\log (2n' + 1) = \log (2n + 1) + A.$$

Putting  $A = 2 \log a$ ,

$$2n' + 1 = (2n + 1)a^2,$$

or we may say

$$n' = a^2 n.$$

We shall then get

$$S' = \log (2a).$$

We might have put  $A = 2 \log [\frac{1}{2}f(n)]$ ,

$$\therefore n' = [\frac{1}{2}f(n)]^2 n$$

and

$$S' = \lim \log [f(n)].$$

As another example take

$$S = 1 - \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n}} + \dots$$

(1) becomes

$$\sqrt{2n' + 1} = \sqrt{2n + 1} + \frac{1}{2}A.$$

Putting  $A = 4a$ , and  $2n + 1 = (2r + 1)^2$ ,

$$n = 2r(r + 1),$$

$$n' = 2(a + r)(a + r + 1),$$

$$n' - n = 2a(a + 2r + 1);$$

$$\Delta n = 4(r + 1),$$

$$\Delta n' = 4(a + r + 1),$$

$$\Delta^2 n = \Delta^2 n' = 4.$$

Hence we have the following system of values :

$$\begin{aligned}
 \text{For } r &= 1, \quad 2, \quad 3, \dots r, \dots, \\
 \Delta n &= 4, \quad 8, \quad 12, \dots 4(r+1), \dots, \\
 n &= 4, \quad 12, \quad 24, \dots 2r(r+1), \dots, \\
 \alpha_{2r} &= \frac{1}{\sqrt{4r(r-1)+2}} + \dots + \frac{1}{\sqrt{4r(r+1)}}. \\
 \Delta n' &= 4(a+1), \quad 4(a+2), \quad 4(a+3), \dots 4(a+r+1), \dots, \\
 n' &= 2(a+1)(a+2), \quad 2(a+2)(a+3), \dots 2(a+r)(a+r+1), \dots, \\
 \alpha_{2r-1} &= \frac{1}{\sqrt{4(a+r)(a+r-1)+1}} + \dots + \frac{1}{\sqrt{4(a+r)(a+r+1)-1}}. \\
 2\sqrt{4(a+r)(a+r+1)+1} &= 2(2a+2r+1), \\
 2\sqrt{4r(r+1)+2} &= 4r+2 + \frac{1}{2r} - \frac{1}{4r^3} + \dots
 \end{aligned}$$

An application of the method employed in this paper will give

$$\begin{aligned}
 \lim \alpha_{2r-1} &= \lim [2(2a+2r+1) - 2(2a+2r-1)] = 4, \\
 \lim \alpha_{2r} &= \lim [2\sqrt{4r(r+1)+2} - 2\sqrt{4r(r-1)+2}] \\
 &= \lim \left( 4 - \frac{1}{2r^3} + \dots \right),
 \end{aligned}$$

therefore

$$\lim (\alpha_{2r-1} - \alpha_{2r}) = \lim \left( \frac{1}{2r^3} - \dots \right).$$

The series

$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \dots$$

in this case will not be convergent but "oscillating." We may, however, mix the positive and negative terms, as suggested above, in such a way as to produce a convergent series having for value  $S + 2a$  without using any parentheses.

$\alpha_{2r-1}$  and  $\alpha_{2r}$  consist of terms of the original series  $\Delta n'$  and  $\Delta n$  in number. Our rule then would be: Take  $a+r+1$  positive terms and distribute among them  $r+1$  negative terms. Do this four times for every value of  $r$ , beginning with zero.

Suppose we take  $a = -1$ . We have

$$S' = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{8}} \\ - \frac{1}{\sqrt{10}} + 1 - \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{16}} - \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{20}} - \frac{1}{\sqrt{22}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{24}} \\ - \dots = S - 2.$$

We can say in regard to the series  $S$ : If  $n' - n$  is a quantity of the same order of magnitude as  $n$ ,  $S'$  will be divergent. If  $n' - n$  is finite,  $S' = S$ . This will be true of any semi-convergent series.

If  $n' - n$  is of the same order of magnitude as the square root of  $n$ ,  $S'$  will usually converge to a value different from  $S$ .

BROWN UNIVERSITY, *October*, 1893.